14.3/14.4 Partial Der. & Tangent Planes

Note: A variable can be treated as

1. A constant (constant term or coef)
2. An independent variable (input)
3. A dependent variable (output)

Entry Task: Find the derivatives

a) $\underset{y(x)}{y} = f(x) = \underbrace{x^2}_{F}\underbrace{e^x}_{S}$

$$\frac{dy}{dx} = \underbrace{x^2}_{F}\,\underbrace{e^x}_{S'} + \underbrace{2x}_{F'}\,\underset{S}{e^x}$$

$$= (x^2 + 2x)\,e^x$$

b) An object's motion (x,y) = (x(t),y(t)) satisfies y = x^2 for all times.

$$\frac{d}{dt}\left(y(t) = (x(t))^2 \right)$$

$$1 \cdot \frac{dy}{dt} = 2x\,\frac{dx}{dt}$$

$$\frac{dy}{dt} = 2x\,\frac{dx}{dt}$$

c) $\overset{z(x)}{z} = x^2 + y^3e^{6y} - 5xy^4 + \ln(w)$

$$\frac{\partial z}{\partial x} = 2x - 5y^4$$

$$\frac{\partial z}{\partial y} = y^3e^{6y}\cdot 6 + 3y^2e^{6y} - 20xy^3$$

$$\frac{\partial z}{\partial w} = \frac{1}{w}$$

d) $x^2 + y^3 = 1,\ \frac{dy}{dx} = ??$ ← output ← input

$\frac{d}{dx}\left(x^2 + (y(x))^3 = 1\right)$

$2x + 3(y(x))^2 y'(x) = 0$

$2x + 3y^2 \frac{dy}{dx} = 0$

$3y^2 \frac{dy}{dx} = -2x$

$\boxed{\frac{dy}{dx} = -\frac{2x}{3y^2}}$

e) $x^2 + t^3 + y^2 - z^2 = 1,\ \frac{\partial z}{\partial x} = ??$

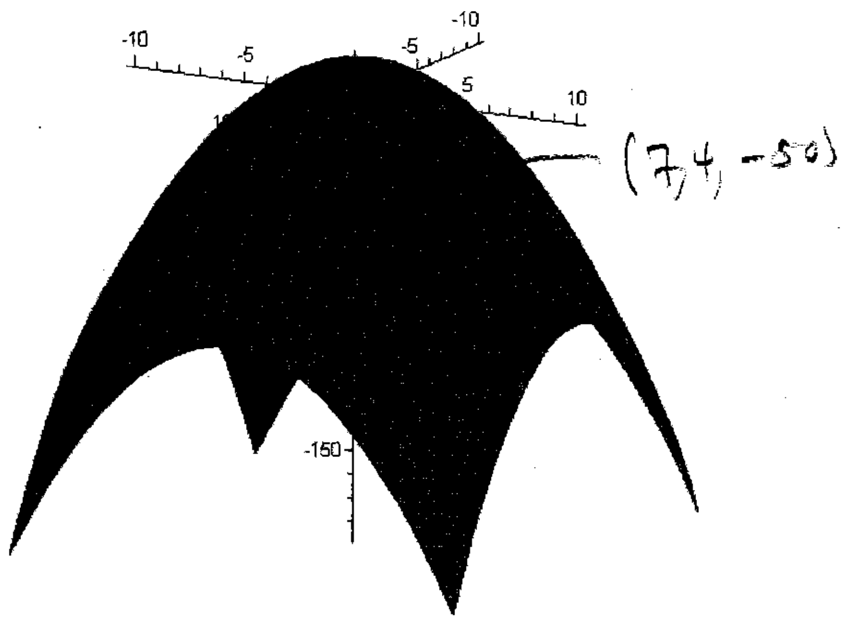
$\frac{\partial}{\partial x}\left[x^2 + t^3 + y^2 - (z(x))^2 = 1\right]$

$2x + 0 + 0 - 2z\frac{\partial z}{\partial x} = 0$

$-2z\frac{\partial z}{\partial x} = -2x$

$\boxed{\frac{\partial z}{\partial x} = \frac{x}{z}}$

Graphical Interpretation of Partial Der:
Pretend you are skiing on the surface $z = f(x,y) = 15 - x^2 - y^2$.



Exercise:

1. Find $f_x(x,y)$ and $f_y(x,y)$

$f_x = -2x$

$f_y = -2y$

2. Assume you are standing on the point on the surface corresponding to (x,y) = (7,4). Compute:
   i) $f(7,4) = 15 - (7)^2 - (4)^2 = -50$
   ii) $f_x(7,4) = -2(7) = -14$
   iii) $f_y(7,4) = -2(4) = -8$

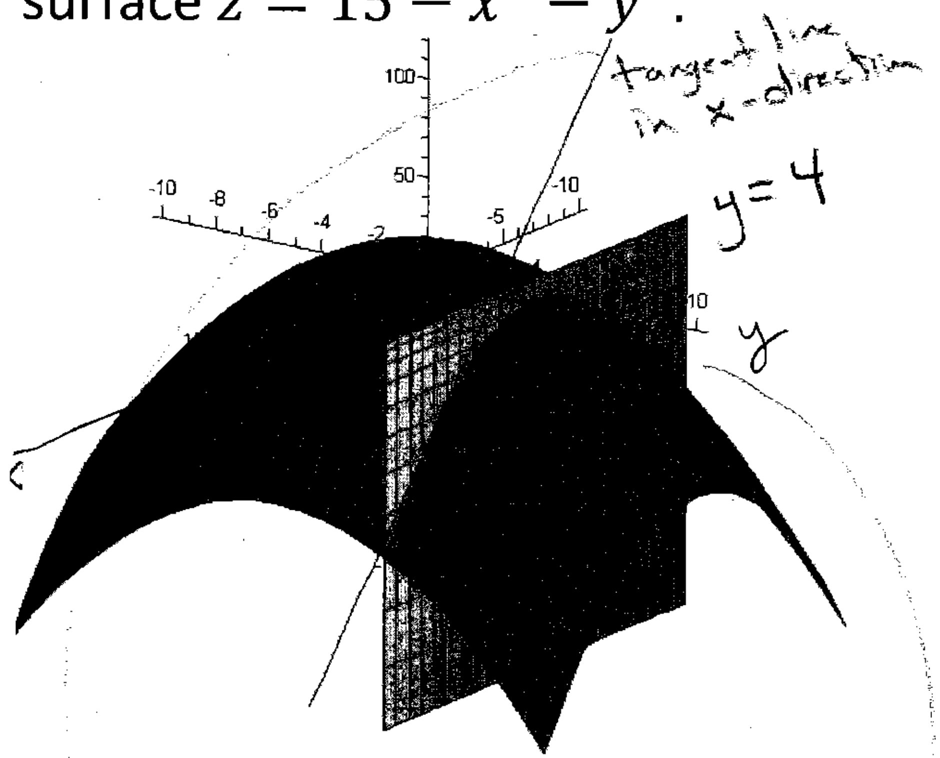
   What do these three numbers represent?

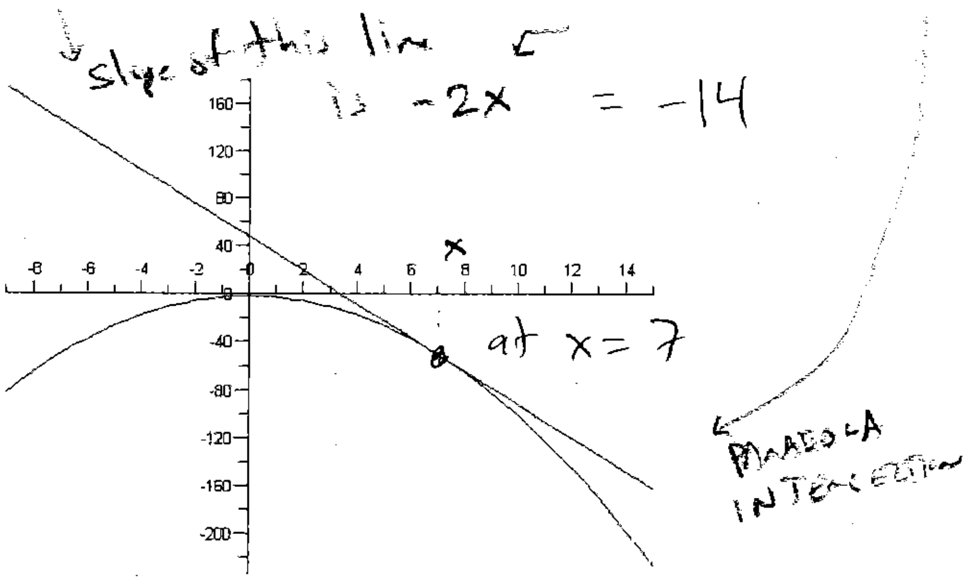
$f(7,4) = -50$ = HEIGHT

$f_x(7,4) = -2(7) = -14$
= "SLOPE IN x-DIRECTION"

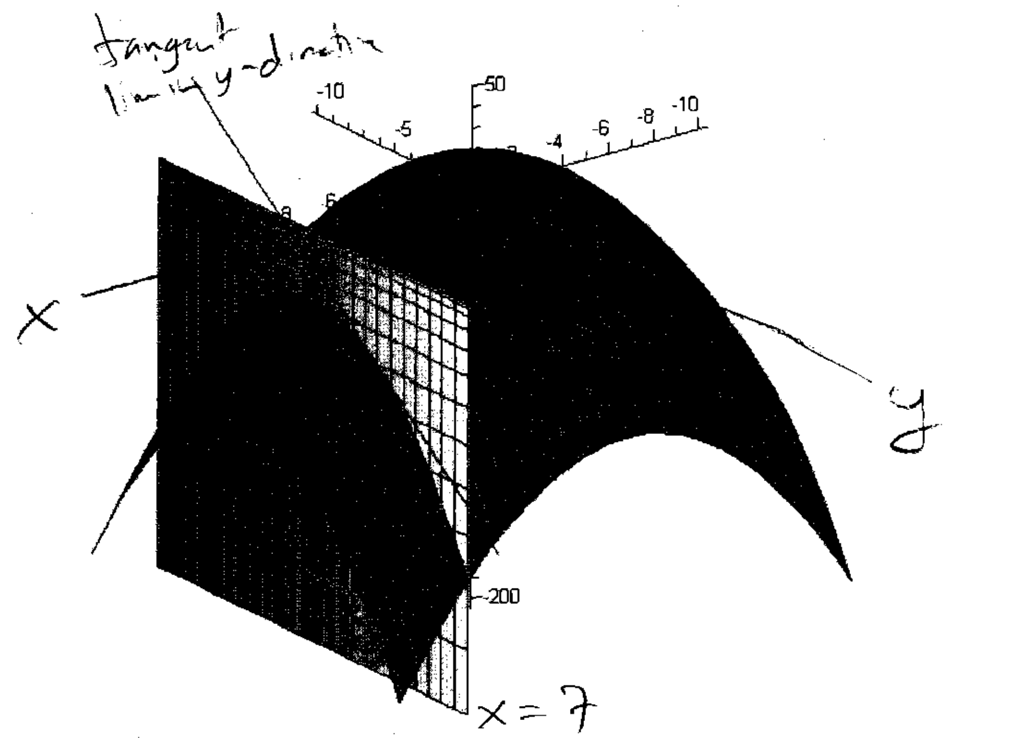
$f_y(7,4) = -2y = -2(4) = -8$
= "SLOPE IN y-DIRECTION"

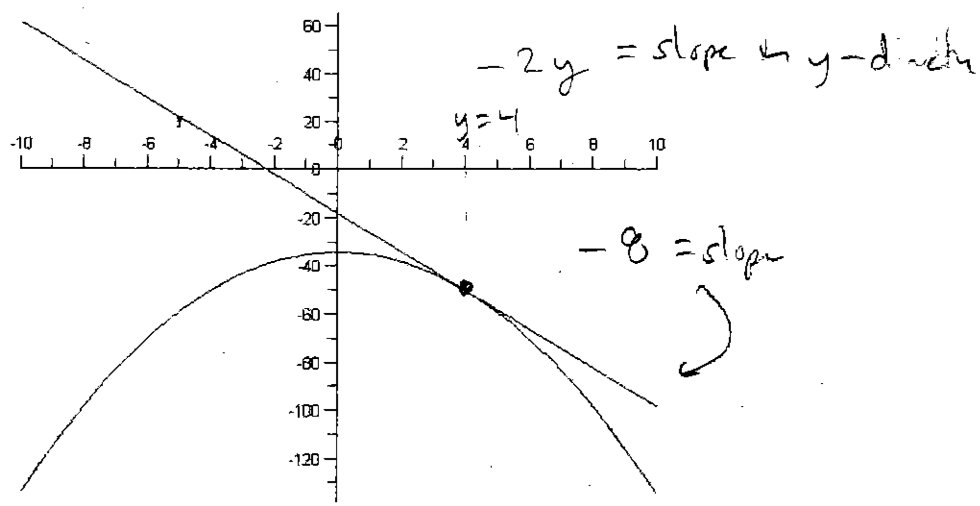
The plane y = 4 intersecting the surface $z = 15 - x^2 - y^2$.





The plane x = 7 intersecting the surface $z = 15 - x^2 - y^2$.





Second Partial Derivatives

Concavity in *x*-direction:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = f_{xx}(x, y)$$

Concavity in *y*-direction:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = f_{yy}(x, y)$$

Mixed Partials:

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = f_{xy}(x, y)$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = f_{yx}(x, y)$$

Example: Find all second partials for

$$z = f(x, y) = x^4 + 3x^2y^3 + y^5$$

$f_x = 4x^3 + 6xy^3$ $\qquad$ $f_y = 9x^2y^2 + 5y^4$

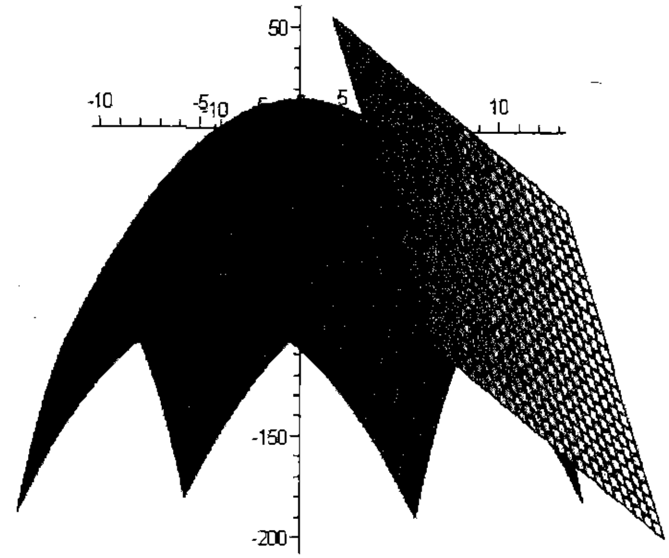
$f_{xx} = 12x^2 + 6y^3$ $\qquad$ $f_{yy} = 18x^2y + 20y^3$

$f_{xy} = 18xy^2 \iff f_{yx} = 18xy^2$

SAME!!!

Always the same.

14.4 Tangent Planes (linear approx.)



The <u>tangent plane</u> to a surface at a point is the plane that contains all tangent lines at that point. It is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane to

$z = f(x,y) = 15 - x^2 - y^2$ at $(7,4)$

$x_0 \quad y_0$

Recall:

$f(4,7) =$ $-50 = z_0$

$f_x(4,7) =$ -14

$f_y(4,7) =$ -8

$$z - -50 = -14(x-7) + -8(y-4)$$

$$\boxed{z + 50 = -14(x-7) - 8(y-4)}$$

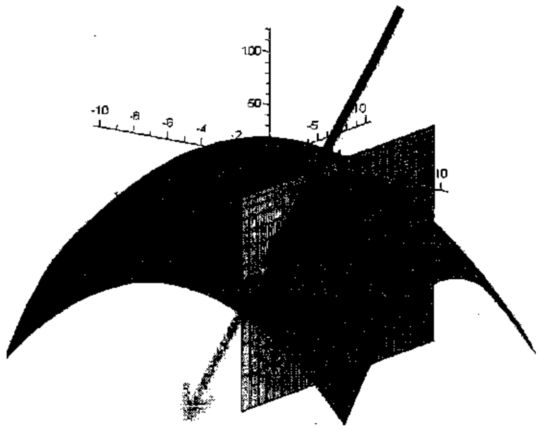
Derivation of Tangent Plane

The plane goes thru (7, 4, -50).
Now we need a normal vector.

Note:

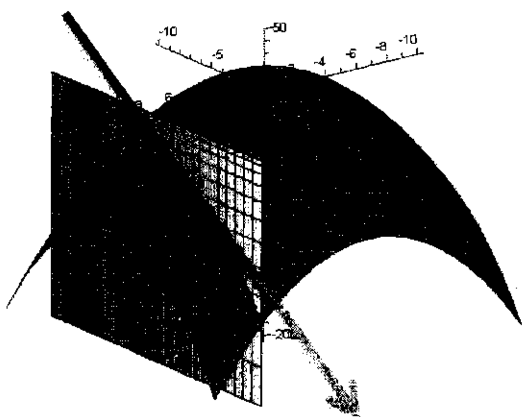
$f_x(x,y) = -2x$

$f_x(7,4) = -14$



$f_y(x,y) = -2y$

$f_y(7,4) = -8$



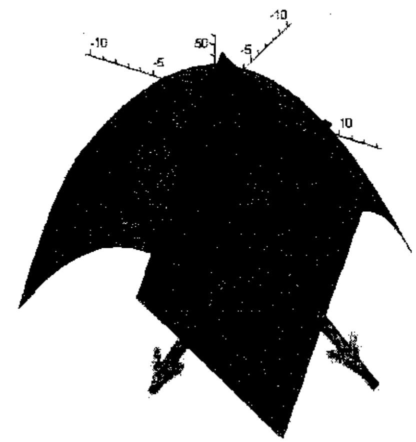
Thus, we can get two vectors that are parallel to the plane:

$$<1,0,f_x(x_0,y_0)> = <1,0,-14>$$

$$<0,1,f_y(x_0,y_0)> = <0,1,-8>$$

So a normal vector is given by

$<1,0,-14> \times <0,1,-8> = <14, 8, 1>$



Tangent Plane:

$$14(x-7) + 8(y-4) + (z+50) = 0$$

Which we rewrite as:

$$z + 50 = -14(x-7) - 8(y-4)$$

General Derivation

In general, for $z = f(x,y)$ at (x_0, y_0) by:

1. $z_0 = f(x_0, y_0)$ = height.

2. $\langle 1,0, f_x(x_0, y_0)\rangle$ = 'a tangent in *x*-dir.'
   $\langle 0,1, f_y(x_0, y_0)\rangle$ = 'a tangent in *y*-dir.'

3. Normal to surface:

$$\langle 1,0, f_x(x_0, y_0)\rangle \times \langle 0,1, f_y(x_0, y_0)\rangle$$
$$= \langle -f_x(x_0, y_0), -f_y(x_0, y_0), 1\rangle$$

Tangent Plane:

$$-f_x(x_0, y_0)(x - x_0) - f_y(x_0, y_0)(y - y_0) + (z - z_0) = 0$$

which we typically write as:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example: Find the tangent plane for

$$f(x,y) = x^2 + 3y^2x - y^3$$

at $(x,y) = (2,1)$.

$f(2,1) = (2)^2 + 3(1)^2(2) - (1)^3 = 4 + 6 - 1 = 9$ = "HEIGHT"

$f_x(x,y) = 2x + 3y^2$ $\quad f_x(2,1) = 2(2) + 3(1)^2 = 7$ = "SLOPE IN x-DIRECTION"

$f_y(x,y) = 6yx - 3y^2$ $\quad f_y(2,1) = 6(1)(2) - 3(1)^2 = 12 - 3 = 9$ = "SLOPE IN y-DIRECTION"

$$\boxed{z - 9 = 7(x-2) + 9(y-1)}$$

An Application of the Tangent Planes

Linear Approximation

"Near" the point (x_0,y_0) the tangent plane and surface z-values are close.

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0),$$

which is the same as

$$L(x, y) = z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Idea:

$$f(x, y) \approx L(x, y) \text{ for } (x, y) \approx (x_0, y_0)$$

Example:

Use the linear approximation to $f(x,y) = x^2 + 3y^2x - y^3$ at $(x,y) = (2,1)$ to estimate the value of $f(1.9, 1.05)$.

WE ALREADY FOUND $z - 9 = 7(x-2) + 9(y-1) \Rightarrow z = \overbrace{9 + 7(x-2) + 9(y-1)}^{L(x,y)}$

SO "NEAR" $(2,1)$ WE HAVE $\boxed{x^2 + 3y^2x - y^3 \approx 9 + 7(x-2) + 9(y-1)}$

AT $x = 1.9$, $y = 1.05$

$L(1.9, 1.05) = 9 + 7(1.9-2) + 9(1.05-1) = 9 - 0.7 + 0.45$
$= 8.75$

ACTUAL $= f(1.9, 1.05) = (1.9)^2 + 3(1.05)^2(1.9) - (1.05)^3 \approx 8.736625$